

1. Title

The mono-anabelian geometry of geometrically pro- p arithmetic fundamental groups of associated low-dimensional configuration spaces I - II - III

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- [Hgsh2], to appear in PRIMS (My doctoral thesis)

Part I 2 July, 2021

Part II 8 July, 2021

Part III 8 July, 2021

URL of this PDF: <https://www.kurims.kyoto-u.ac.jp/~higashi/20210702.pdf>

2. Abstract

Let p be a prime number. In this talk, we study geometrically pro- p arithmetic fundamental groups of low-dimensional configuration spaces associated to a given hyperbolic curve over an arithmetic field such as a number field or a p -adic local field. Our main results concern the group-theoretic reconstruction of the function field of certain tripods (i.e., copies of the projective line minus three points) that lie inside such a configuration space from the associated geometrically pro- p arithmetic fundamental group, equipped with the auxiliary data constituted by the collection of decomposition groups determined by the closed points of the associated compactified configuration space.

3. Part I

We explain the statement of Mochizuki's results and today's main results. Finally, we explain "From PGCS to CFS" in the proof of main results. By this explanation, we confirm what kind of operation mono-anabelian geometry.

- What is Grothendieck Conjecture (p.5)
- Various GC (p.6-12)
- Main Theorem and flowchart of reconstruction (p.13-14)
- From PGCS to CFS (p.16-21)

4. Part II, Part III

We explain the idea of Main Theorem in comparison with the past results. After that, we explain Main Theorems as possible as.

- Review of Part I
- Idea (p.23-26)
- From CFS to base fields (p.28-38)
- From trip. very ample (g, r, n) PGCS to $(0, 3, 2)$ PGCS (p.39-46)
- Semi-absolute bi pro- p GC (p.47-49)
- Reconstruct function fields (p.50-54)

5. What is Grothendieck Conjecture

What is Grothendieck conjecture (GC): omit

k : a suitable field

${}^\dagger U, {}^\ddagger U$: suitable varieties

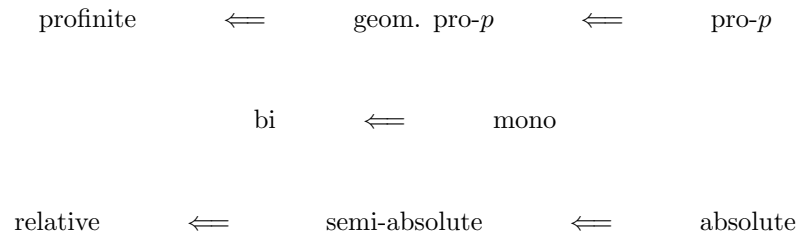
Then is the natural map

$$\text{Isom}_k({}^\dagger U, {}^\ddagger U) \rightarrow \text{Isom}_{G_k}(\pi_1({}^\dagger U), \pi_1({}^\ddagger U)) / \text{Inn}$$

bijjective?

6. various GC

various choice



• Past results

Mochizuki, [Topics], Theorem 4.12: relative bi geom. pro- Σ GC

Mochizuki, [AbsTpIII], Corollary 1.10, (absolute) mono profinite GC

⋮

• Today's main results

Higashiyama, [Hgsh2], Theorem 0.1, semi-absolute bi pro- p GCHigashiyama, [Hgsh2], Theorem 0.2, (semi-absolute) mono pro- p GC

7. Notation

p : a prime number

n : a positive integer

k : a generalized sub- p -adic field

$(\stackrel{\text{def}}{\iff} k \stackrel{\exists}{\hookrightarrow} \text{finitely generated } / (\mathbb{Q}_p^{\text{unr}})^{\wedge})$

\bar{k} : the algebraic closure

X^{log}/k : an ordered hyperbolic log curve of type (g, r) ($2g - 2 + r > 0$)

T^{log}/k : an ordered hyperbolic log curve of type $(0, 3)$

X_n^{log} : the n -th log configuration space of type (g, r, n)

For any log scheme S^{log} , write S for the underlying scheme of S^{log}

write U_S for the interior of S^{log}

In particular, X is a proper curve of type $(g, 0)$

U_X is a hyperbolic curve of type (g, r)

X^{log} consists of the hyperbolic curve U_X and r -th log points

X_n^{log} consists of the n -th configuration space U_{X_n} and log divisors

$\pi_1(U_X)$: the étale fundamental group

$G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$

8. Detail: Exact sequence

We consider

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k}) & \longrightarrow & \pi_1(U_{X_n}) & \longrightarrow & G_k \longrightarrow 1 & \text{(profinite)} \\
 & & \downarrow & \searrow \square & \downarrow & & \parallel & \\
 1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{[p]} & \longrightarrow & G_k \longrightarrow 1 & \text{(geom. pro-}p\text{)} \\
 & & \parallel & & \downarrow & & \downarrow & \\
 & & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{(p)} & \longrightarrow & G_k^{(p)} \longrightarrow 1 & \text{(pro-}p\text{)}
 \end{array}$$

When we consider the pro- p exact sequence, we assume that

$$\pi_1(U_{X_n} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{X_n})^{(p)}$$

Note that

$$\pi_1(U_{X_n} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{X_n})^{(p)} \implies \zeta_p \in k$$

In general,

$$\text{profinite} \quad \longleftarrow \quad \text{geom. pro-}p \quad \longleftarrow \quad \text{pro-}p$$

9. Detail: bi? mono?-1

Lemma

If k : a p -adic local field, then by local class field theory, we have the local reciprocity map $\rho_k: k^\times \hookrightarrow G_k^{\text{ab}}$ which induces an isom. $\widehat{k^\times} \xrightarrow{\sim} G_k^{\text{ab}}$

magenta color is an abstract group-like object

Definition(mono)

$$G_k \rightsquigarrow \widehat{k^\times}$$

means

Let G_k be a profinite group which is isom. to G_k

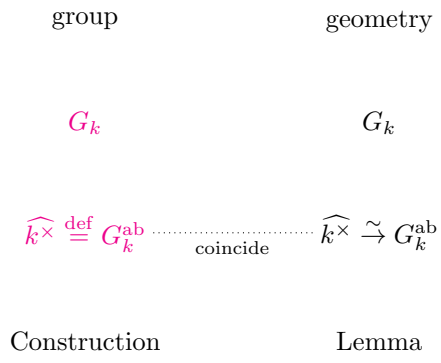
$\widehat{k^\times} \stackrel{\text{def}}{=} G_k^{\text{ab}}$ (only **magenta object** and group-theoretic operation)

Then any isom. $G_k \xrightarrow{\sim} G_k$ induces a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\sim} & G_k \\ \downarrow & & \downarrow \\ \widehat{k^\times} & \xrightarrow{\sim} & \widehat{k^\times} \end{array}$$

10. Detail: bi? mono?-2

In particular, assertion $G_k \rightsquigarrow \widehat{k^\times}$ follows immediately from Lemma and Construction



11. Detail: bi? mono?-3

Proposition(mono-reconstruction)

$$G_k \rightsquigarrow \widehat{k^\times}$$

(pf) It follows from Lemma and Construction

Proposition(bi-reconstruction)

Any isom. $G_k \xrightarrow{\sim} G_k$ induces a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\sim} & G_k \\ \downarrow & & \downarrow \\ \widehat{k^\times} & \xrightarrow{\sim} & \widehat{k^\times} \end{array}$$

(pf) It follows from Lemma

In general,

$$\text{bi} \quad \longleftarrow \quad \text{mono}$$

bi GC

$$\text{Isom}_k(U_{\dagger X}, U_{\ddagger X}) \rightarrow \text{Isom}_{G_k}(\pi_1(U_{\dagger X}), \pi_1(U_{\ddagger X}))/\text{Inn}$$

is bijective?

mono GC

$$\pi_1(U_X) \rightsquigarrow \text{Funct}(U_X) \text{ or } U_X?$$

12. Detail: relative? semi-absolute? absolute?

- relative: Is the morphism

$$\mathrm{Isom}_k(U_X, U_X) \rightarrow \mathrm{Isom}_{G_k}(\pi_1(U_X), \pi_1(U_X)) / \mathrm{Inn}$$

bijjective? i.e.,

$$U_X \xrightarrow{\sim} U_X \text{ vs. } \begin{array}{ccc} \pi_1(U_X) & \longrightarrow & \pi_1(U_X) \\ & \searrow & \swarrow \\ & G_k & \end{array}$$

- semi-absolute:

$$U_X \xrightarrow{\sim} U_X \text{ vs. } \begin{array}{ccc} \pi_1(U_X) & \longrightarrow & \pi_1(U_X) \\ \downarrow & & \downarrow \\ G_k & \longrightarrow & G_k \end{array}$$

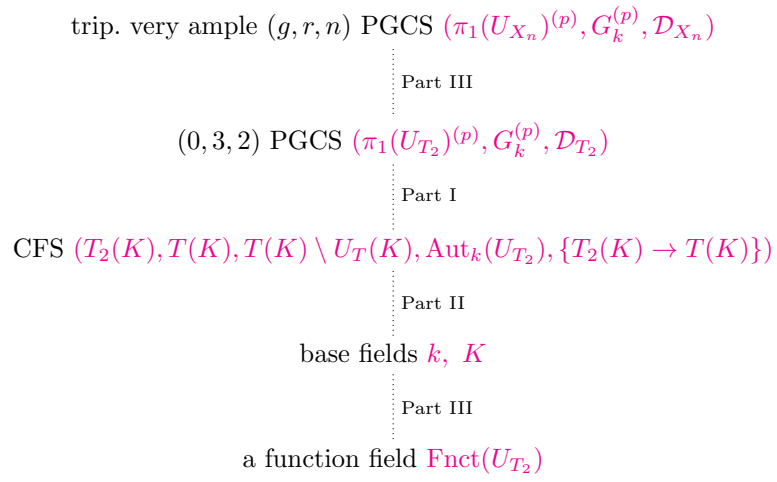
- absolute:

$$U_X \xrightarrow{\sim} U_X \text{ vs. } \pi_1(U_X) \xrightarrow{\sim} \pi_1(U_X)$$

In general,

$$\text{relative} \quad \Leftarrow \quad \text{semi-absolute} \quad \Leftarrow \quad \text{absolute}$$

13. Main Theorem and flowchart of reconstruction-1



14. Main Theorem and flowchart of reconstruction-2

Theorem 0.1 (Part III) semi-abs. bi pro- p GC

$$\text{Isom}(U_{\dagger X_n}, U_{\ddagger X_n}) \xrightarrow{\sim} \text{Isom}(\dagger\text{PGCS}, \ddagger\text{PGCS})/\text{Inn}$$

Theorem 0.2 (Part III) mono pro- p GC

$$\text{trip. very ample PGCS } (\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n}) \rightsquigarrow \text{Funct}(U_{T_2})$$

Theorem 0.3 (Part I; Part II)

$$(0, 3, 2) \text{ PGCS } (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \rightsquigarrow \text{CFS} \rightsquigarrow k, K$$

Theorem 0.4 (Part III)

$$\begin{array}{ccc} (0, 3, 2) \text{ PGCS } (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) & \xrightarrow{\rightsquigarrow} & k \\ & \text{Thm 0.3} & \vdots \text{ inclusion} \\ \pi_1(U_T) & \xrightarrow{\rightsquigarrow} & k \subseteq \text{Funct}(U_T) \\ & [\text{AbsTpIII}], 1.10 & \end{array}$$

15. Comparison

- Past results

Mochizuki, [Topics], Theorem 4.12: relative bi geom. pro- Σ GC

$$\mathrm{Isom}_k(U_{\dagger X}, U_{\ddagger X}) \rightarrow \mathrm{Isom}_{G_k^\Sigma}(\pi_1(U_{\dagger X})^\Sigma, \pi_1(U_{\ddagger X})^\Sigma) / \mathrm{Inn}$$

Mochizuki, [AbsTpIII], Corollary 1.10, (absolute) mono profinite GC

$$\pi_1(U_X) \rightsquigarrow \mathrm{Funct}(U_X)$$

- Today's main results

Higashiyama, [Hgsh2], Theorem 0.1, semi-absolute bi pro- p GC

$$\mathrm{Isom}(U_{\dagger X_n}, U_{\ddagger X_n}) \rightarrow \mathrm{Isom}({}^{\dagger}\mathrm{PGCS}, {}^{\ddagger}\mathrm{PGCS}) / \mathrm{Inn}$$

Higashiyama, [Hgsh2], Theorem 0.2, (semi-absolute) mono pro- p GC

trip. very ample PGCS $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n}) \rightsquigarrow \mathrm{Funct}(U_{T_2})$

16. From PGCS to CFS-1

Definition

k : a generalized sub- p -adic field

$k \subseteq K \subseteq \bar{k}$: the maximal pro- p subextension of \bar{k}/k

X^{\log}/k : an ordered hyperbolic log curve of type (g, r) ($2g - 2 + r > 0$) (cf.

Notation)

Suppose that

$$1 \longrightarrow \pi_1(U_{X_n} \times_k \bar{k})^{(p)} \longrightarrow \pi_1(U_{X_n})^{(p)} \longrightarrow G_k^{(p)} \longrightarrow 1$$

$\pi_1(U_{X_n})^{(p)}$: a profinite group

$G_k^{(p)}$: a quotient group of $\pi_1(U_{X_n})^{(p)}$

\mathcal{D}_{X_n} : a set of subgroups of $\pi_1(U_{X_n})^{(p)}$

We shall refer to $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$ as a (g, r, n) PGCS-collection if there exists a collection of data as follows:

an isomorphism

$$\gamma: \pi_1(U_{X_n})^{(p)} \xrightarrow{\sim} \pi_1(U_{X_n})^{(p)}$$

such that γ induces a commutative diagram

$$\begin{array}{ccc} \pi_1(U_{X_n})^{(p)} & \xrightarrow{\sim} & \pi_1(U_{X_n})^{(p)} \\ \downarrow & \circlearrowleft & \downarrow \\ G_k^{(p)} & \xrightarrow{\sim} & G_k^{(p)} \end{array}$$

and a bijection

$$\mathcal{D}_{X_n} \xrightarrow{\sim} \mathcal{D}_{X_n} \stackrel{\text{def}}{=} \{D \subseteq \pi_1(U_{X_n})^{(p)} \mid D \text{ is a decomposition group associated to some } x \in X_n(K)\}.$$

17. From PGCS to CFS-2

START: $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$$\pi_1(U_{T_2} \times_k \bar{k})^{(p)} \stackrel{\text{def}}{=} \text{Ker}(\pi_1(U_{T_2})^{(p)} \rightarrow G_k^{(p)})$$

$$\text{LFS} \stackrel{\text{def}}{=} \{\text{log-full subgroups of } \pi_1(U_{T_2} \times_k \bar{k})^{(p)}\}.$$

$$\text{LFS} \stackrel{\text{def}}{=} \{D \cap \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \mid D \in \mathcal{D}_{T_2}, D \cap \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \xrightarrow{\sim} \mathbb{Z}_p^{\oplus 2}\}$$

$$\text{GFS} \stackrel{\text{def}}{=} \{\text{generalized fiber subgroups of } \pi_1(U_{T_2} \times_k \bar{k})^{(p)}\}$$

$$\text{For } F \in \text{GFS}, \pi_1(U_{T_2} \times_k \bar{k})^{(p)} / F \xrightarrow{\sim} \pi_1(U_T \times_k \bar{k})^{(p)}$$

Proposition

$$(0, 3, 2) \text{ PGCS-collection } (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \rightsquigarrow \text{GFS}$$

(pf.) There exist two reconstruction.

- [HMM], Theorem 2.5, (iv),
- [Hgsh3], Theorem A, (v), (using LFS)

18. From PGCS to CFS-3

Fix $E \in \text{GFS}$

$$\pi_1(U_T \times_k \bar{k})^{(p)} \stackrel{\text{def}}{=} \pi_1(U_{T_2} \times_k \bar{k})^{(p)} / E$$

$$\pi_1(U_T)^{(p)} \stackrel{\text{def}}{=} \pi_1(U_{T_2})^{(p)} / E$$

$p_{2/1}^\pi: \pi_1(U_{T_2})^{(p)} \rightarrow \pi_1(U_T)^{(p)}$: the natural quotient homomorphism.

$$T_2(K) \stackrel{\text{def}}{=} \{\pi_1(U_{T_2} \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups } \in \mathcal{D}_{T_2}\}$$

where, $k \subseteq K \subseteq \bar{k}$: the maximal pro- p subextension of \bar{k}/k

i.e., $G_k^{(p)} = \text{Gal}(K/k)$

$$\mathcal{D}_T \stackrel{\text{def}}{=} \{C_{\pi_1(U_T)^{(p)}}(p_{2/1}^\pi(D)) \mid D \in \mathcal{D}_{T_2}\}$$

where $C_{\pi_1(U_T)^{(p)}}(-)$ denotes the commensurator of $(-)$ in $\pi_1(U_T)^{(p)}$

$$T(K) \stackrel{\text{def}}{=} \{\pi_1(U_T \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups } \in \mathcal{D}_T\}$$

$$T(K) \setminus U_T(K) \stackrel{\text{def}}{=} \{[D] \in T(K) \mid D \cap \pi_1(U_T \times_k \bar{k})^{(p)} \neq \{1\}\} \subseteq T(K)$$

where $T(K) \setminus U_T(K) = \{0, 1, \infty\}$

19. From PGCS to CFS-4

$$\mathrm{Aut}_k(U_{T_2}) \subseteq \mathrm{Aut}(T_2(K))$$

for the group of bijections $T_2(K) \xrightarrow{\sim} T_2(K)$ induced by the group $\mathrm{Out}_{G_k^{(p)}}(\pi_1(U_{T_2})^{(p)})$ of $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$ -outer automorphisms of the profinite group $\pi_1(U_{T_2})^{(p)}$ lying over the identity automorphism of $G_k^{(p)}$

$$p_{2/1}^\pi: \pi_1(U_{T_2})^{(p)} \rightarrow \pi_1(U_T)^{(p)} \text{ induces } p_{2/1}^T: T_2(K) \rightarrow T(K) \\ \{T_2(K) \rightarrow T(K)\} \text{ for the } \mathrm{Aut}_k(U_{T_2})\text{-orbit of } p_{2/1}^T$$

We construct CFS-collection

$$(T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$$

20. From PGCS to CFS-5

Key Lemma

$$T_2(K) \xrightarrow{\sim} \{\pi_1(U_{T_2} \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups} \in \mathcal{D}_{T_2}\}$$

Surj: immediately

Inj: By pro- p version of [Topics], Theorem 4.12 (relative bi)

Key Lemma

$$\text{Aut}_k(U_{T_2}) \xrightarrow{\sim} \text{Out}_{G_k^{(p)}}(\pi_1(U_{T_2})^{(p)})$$

By pro- p version of [Topics], Theorem 4.12 (relative bi)

21. From PGCS to CFS-6

Theorem

$(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$
 \rightsquigarrow CFS-collection $(T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$

Not done yet

$\rightsquigarrow 0, 1, \infty \in T(K) \setminus U_T(K) \quad (T(K) \setminus U_T(K) = \{0, 1, \infty\})$
 $\rightsquigarrow T(K) \setminus \{\infty\}$: additive law and mult. law $(T(K) \setminus \{\infty\} = \mathbb{P}^1(K) \setminus \{\infty\} = K)$

Continue to Part II

22. Idea-1: Highlights of mono-anabelian geometry

The highlight is to reconstruct a base field k

There exists a few result

Mochizuki, [AbsTpIII], Corollary 1.10, mono profinite GC

More details: 1 September, 2021

START: $\pi_1(U_X)$

- $\rightsquigarrow \pi_1(U_X) \rightarrow G_k$ ([AbsTpI], Corollary 2.8)
- \rightsquigarrow the set X (Belyi cuspidalization)
- $\rightsquigarrow k^\times \subseteq \widehat{k^\times} \stackrel{\text{def}}{=} H^1(G_k, \widehat{\mathbb{Z}}(1))$, and k^\times : multi. law (Kummer theory)
- $\rightsquigarrow k^\times, \text{Div}(X) \subseteq \oplus H^1$: additive law (Uchida's Lemma)

23. Idea-2

Mochizuki's result is mono profinite GC

I want to consider the pro- p version

But...

Try to update to the pro- p version

START: $\pi_1(U_X)^{(p)}$

• reconstruct $G_k^{(p)}$: can't

(Note that $\pi_1(U_X)^{(p)} \rightsquigarrow G_k^{(p)}$, by Tsujimura)

• using Belyi cuspidalization: can't

• using Kummer theory: can't

In particular, $\not\rightsquigarrow k^\times$ and mult. law

24. Idea-3: Why we start from PGCS?

Today's theorem

START: $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$\rightsquigarrow (T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$ (by Part I)

- $G_k^{(p)}$: assumption

- $\rightsquigarrow T(K)$ (using \mathcal{D}_{T_2})

where $T(K) = K \sqcup \{\infty\}$ (the curve vs. the base field)

(cf. Mochizuki, [AbsTpII], Corollary 2.9, absolute bi profinite GC equipped with decomposition groups)

25. Idea-4

additive law? mult. law?

Using curves

$\text{Aut}_k(U_T) \xrightarrow{\sim} S_3$ (small)

For $a \in U_T$, $a \mapsto a, \frac{1}{a}, 1 - a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}$

Think of higher dimension!

Using second configuration spaces

$\text{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$ (big)

For $(a, b) \in U_{T_2}$, $(a, b) \mapsto (\frac{1}{a}, \frac{b}{a}), (\frac{b-a}{b}, \frac{b-a}{b-1}), \dots$

Construct additive law and mult. law!

(cf. $G_k \not\rightsquigarrow k$, but $\pi_1(U_X) \rightsquigarrow k$ by Mochizuki, [AbsTpIII], Corollary 1.10)

26. Idea5: Summary

$$\pi_1(U_T)^{(p)} \overset{\text{?}}{\dashrightarrow} G_k^{(p)}, T(K)$$

$$(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \overset{\sim}{\dashrightarrow} G_k^{(p)}, T(K)$$

Why genus 0 curve? \implies the curve vs. the base field ($T(K) = K \sqcup \{\infty\}$)

Why configuration spaces? \implies If we use GC for curves, configuration spaces will be appropriate

Why (0,3,2)? (second configuration spaces of (0,3) curves) \implies

- the set of automorphisms $\text{Aut}_k(U_{T_2}) (\overset{\sim}{\rightarrow} S_5)$ is bigger than others $\text{Aut}_k(U_{X_2}) (\overset{\sim}{\rightarrow} S_2)$
- (0,3,2) configuration spaces are contained in most configuration spaces (i.e., trip. very ample)

\implies START (0,3,2) PGCS $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

27. Relationship between configuration space T_2 and $0, 1, \infty \in T$

$$\begin{array}{ccc}
 T_2 & \xrightarrow{\sim} & \overline{\mathcal{M}}_{0,5} & (x, y) \cdots \cdots (0, 1, \infty, x, y) \\
 p_2^T \downarrow & & p_5 \downarrow & \vdots \qquad \qquad \qquad \vdots \\
 T & \xrightarrow{\sim} & \overline{\mathcal{M}}_{0,4} & x \cdots \cdots (0, 1, \infty, x)
 \end{array}$$

Proposition

Let $x \in T(K) \setminus U_T(K)$ ($= \{0, 1, \infty\}$)

- $x = 0 \iff x = p_5 \circ (2\ 3)(x, y)$ for $y \in U_T(K)$
 - $x = 1 \iff x = p_5 \circ (1\ 3)(x, y)$ for $y \in U_T(K)$
 - $x = \infty \iff x = p_5 \circ (1\ 2)(x, y)$ for $y \in U_T(K)$
- where, $(2\ 3), (1\ 3), (1\ 2) \in S_5$ and $(x, y) \in T_2(K)$

Relationship between configuration space T_2 and $0, 1, \infty \in T$ is determined by the labeling $S_5 \xrightarrow{\sim} \text{Aut}_k(U_{T_2}) \curvearrowright T_2$ and projections p_1, \dots, p_5

28. From CFS to base fields-1

START:

CFS-collection $(T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$ For $\dagger\lambda, \ddagger\lambda \in \{T_2(K) \rightarrow T(K)\}$

$$\dagger\lambda \sim \ddagger\lambda \stackrel{\text{def}}{\iff} \{\dagger\lambda^{-1}(b)\}_{b \in T(K) \setminus U_T(K)} = \{\ddagger\lambda^{-1}(b)\}_{b \in T(K) \setminus U_T(K)}$$

CFS $\rightsquigarrow \{T_2(K) \rightarrow T(K)\}_\sim$ Note that $S_5 \xrightarrow{\sim} \text{Aut}_k(U_{T_2}) \curvearrowright T_2(K)$, $S_3 \xrightarrow{\sim} \text{Aut}_k(U_T) \curvearrowright T(K)$ $\#\{T_2(K) \rightarrow T(K)\} = 30$, $\{T_2(K) \rightarrow T(K)\} = \bigsqcup S_3 \circ p_i$, $\#\{T_2(K) \rightarrow T(K)\}_\sim = 5$, $\{T_2(K) \rightarrow T(K)\}_\sim = \{S_3 \circ p_i\}$ Let $\phi: \text{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$: fix any isom.Write $\{T_2(K) \rightarrow T(K)\}_a \in \{T_2(K) \rightarrow T(K)\}_\sim$ for the unique class s.t.

$$\{T_2(K) \rightarrow T(K)\}_a = \{T_2(K) \rightarrow T(K)\}_a \circ (\phi^{-1}(b \ c))$$

for all transpositions $(b \ c) \in S_5$ s.t. $a \notin \{b, c\}$ If $\phi: \text{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$: canonical, then $\{T_2(K) \rightarrow T(K)\}_i = S_3 \circ p_i \ni p_i$

29. From CFS to base fields-2

Let $\lambda \in \{T_2(K) \rightarrow T(K)\}_1$: fix

$$p_2 \stackrel{\text{def}}{=} \lambda \circ (\phi^{-1}(1\ 2)), \quad p_i \stackrel{\text{def}}{=} p_{i-1} \circ (\phi^{-1}(i-1\ i))$$

Let $x, y \in T(K)$ be distinct elements. Write $(x, y) \in T_2(K)$ for the unique element such that $p_5(x, y) = x, p_4(x, y) = y$

Write **0** for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(2\ 3))(x, y)$ for every $y \in T(K) \setminus \{x\}$

Write **1** for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(1\ 3))(x, y)$ for every $y \in T(K) \setminus \{x\}$

Write ∞ for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(1\ 2))(x, y)$ for every $y \in T(K) \setminus \{x\}$

Relationship between configuration space T_2 and $0, 1, \infty \in T$ is determined by the labeling $S_5 \xrightarrow{\sim} \text{Aut}_k(U_{T_2}) \curvearrowright T_2$ and projections p_1, \dots, p_5 (cf. p.27)

30. From CFS to base fields-3

- We write

$$\tau_{\text{rf}} \stackrel{\text{def}}{=} \phi^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

- We write

$$\tau_{\text{ra}} \stackrel{\text{def}}{=} \phi^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

- We write

$$\tau_{\text{cr}} \stackrel{\text{def}}{=} \phi^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Note that if $\phi: \text{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$: canonical then

$$\tau_{\text{rf}}: (x, y) \mapsto (0, 1, \infty, x, y) \xrightarrow{\tau_{\text{rf}}} (0, 1, x, \infty, y) \xrightarrow{\frac{t(1-x)}{t-x}} \left(0, 1, \infty, 1-x, \frac{y(x-1)}{x-y}\right) \mapsto \left(1-x, \frac{y(x-1)}{x-y}\right)$$

$$\tau_{\text{ra}}: (x, y) \mapsto (0, 1, \infty, x, y) \xrightarrow{\tau_{\text{ra}}} (0, x, \infty, 1, y) \xrightarrow{\frac{t}{x}} \left(0, 1, \infty, \frac{1}{x}, \frac{y}{x}\right) \mapsto \left(\frac{1}{x}, \frac{y}{x}\right)$$

$$\tau_{\text{cr}}: (x, y) \mapsto (0, 1, \infty, x, y) \xrightarrow{\tau_{\text{cr}}} (\infty, x, y, 0, 1) \xrightarrow{\frac{y-x}{y-t}} \left(0, 1, \infty, \frac{y-x}{y}, \frac{y-x}{y-1}\right) \mapsto \left(\frac{y-x}{y}, \frac{y-x}{y-1}\right)$$

31. From CFS to base fields-4

We shall say that a collection of maps

$$\begin{aligned} +, \times &: (T(K) \setminus \{\infty\}) \times (T(K) \setminus \{\infty\}) \rightarrow (T(K) \setminus \{\infty\}) \\ - &: (T(K) \setminus \{\infty\}) \rightarrow (T(K) \setminus \{\infty\}) \\ / &: (T(K) \setminus \{0, \infty\}) \rightarrow (T(K) \setminus \{0, \infty\}) \end{aligned}$$

is CFS-admissible if the following conditions are satisfied:

$$-0 = 0, \quad /1 = 1$$

For $x, y \in T(K) \setminus \{\infty\}$, $x + y = y + x$, $x \times y = y \times x$

For $x \in T(K) \setminus \{\infty\}$, $0 + x = x$, $0 \times x = 0$, $1 \times x = x$, $x + (-x) = 0$

For $x \in T(K) \setminus \{0, \infty\}$, $x \times (/x) = 1$

Let $x, y \in U_T(K)$ s.t. $x \neq y$. Then $/x = p_5(\tau_{\text{ra}}(x, y))$

Let $x, y \in U_T(K)$ s.t. $y \neq /x$. Then $x \times y = p_4(\tau_{\text{ra}}(/x, y))$

Note that

$$\begin{aligned} (x, y) &\xrightarrow{\tau_{\text{ra}}} \left(\frac{1}{x}, \frac{y}{x} \right) \xrightarrow{p_5} \frac{1}{x} \\ \left(\frac{1}{x}, y \right) &\xrightarrow{\tau_{\text{ra}}} (x, xy) \xrightarrow{p_4} xy \end{aligned}$$

32. From CFS to base fields-5

There exists an element $x \in U_T(K)$ s.t. $/x = x$

Let $x, y \in U_T(K)$ s.t. $/x = x$. Then $-1 = x \in T(K)$, $-y = x \times y$

Let $x \in U_T(K) \setminus \{-1\}$. Then $1 + 1 = p_5(\tau_{\text{rf}}(-1, x))$

Let $x \in U_T(K) \setminus \{-1\}$. Then $x + 1 = p_5(\tau_{\text{cr}}(x, -1))$

Let $x \in T(K) \setminus \{\infty\}$ s.t. $y \neq 0$. Then $x + y = y \times (\frac{x}{y} + 1)$

Note that

$$\begin{aligned} (-1, x) &\xrightarrow{\tau_{\text{rf}}} \left(1 - (-1), \frac{x(-1-1)}{-1-x}\right) \xrightarrow{p_5} 1 - (-1) = 2 \\ (x, -1) &\xrightarrow{\tau_{\text{cr}}} \left(\frac{-1-x}{-1}, \frac{-1-x}{-1-1}\right) \xrightarrow{p_5} \frac{-1-x}{-1} = x + 1 \end{aligned}$$

If one fixes the data $(\text{CFS}, \phi, \lambda)$, then any CFS-admissible collection of maps is unique

Thus, if the data $(\text{CFS}, \phi, \lambda)$ admits a CFS-admissible collection of maps, then we shall write

$$K = (T(K) \setminus \{\infty\}, +, \times, -, /)$$

for the set $T(K) \setminus \{\infty\}$ equipped with the maps $+, \times, -, /$

33. From CFS to base fields-6: Main Theorems

START: $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$\gamma: (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \xrightarrow{\sim} (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$: any isom.

By Part I, PGCS \rightsquigarrow CFS

In particular, γ induces

$$\begin{aligned} & (T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\}) \\ & \xrightarrow{\sim} (T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\}) \end{aligned}$$

Theorem 3.13, (i)

If $\phi: \text{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$, $\lambda \in \{T_2(K) \rightarrow T(K)\}_1$: good choices

i.e.,

$$\begin{array}{ccc} \text{Aut}_k(U_{T_2}) & \xrightarrow{\sim} & \text{Aut}_k(U_{T_2}) \\ & \searrow \phi & \downarrow \text{canonical} \\ & & S_5 \end{array}$$

$$\begin{aligned} \{T_2(K) \rightarrow T(K)\}_1 & \xrightarrow{\sim} \{T_2(K) \rightarrow T(K)\}_1 = S_3 \circ p_1 \\ & \lambda \mapsto p_1 \end{aligned}$$

Then

$$T(K) \xrightarrow{\sim} T(K) = K \sqcup \{\infty\}$$

induces

$$0, 1, \infty \mapsto 0, 1, \infty,$$

and the field structure of K determines a CFS-admissible collection of maps.

In particular, K may be regarded as a field

34. From CFS to base fields-7

DefinitionLet $\square \in \{\dagger, \ddagger\}$ $\square \mathcal{A} = (\square T_2(K), \square T(K), \square(T(K) \setminus U_T(K)), \square \text{Aut}_k(U_{T_2}), \square \{T_2(K) \rightarrow T(K)\})$:

CFS-collections

 $(\alpha, \beta): \dagger \mathcal{A} \xrightarrow{\sim} \ddagger \mathcal{A}$: an isom. of CFS $\xLeftrightarrow{\text{def}} \alpha: \dagger T_2(K) \xrightarrow{\sim} \ddagger T_2(K), \beta: \dagger T(K) \xrightarrow{\sim} \ddagger T(K)$: bijections of sets s.t. $\beta(\dagger(T(K) \setminus U_T(K))) = \ddagger(T(K) \setminus U_T(K))$ $\alpha \circ \dagger \text{Aut}_k(U_{T_2}) \circ \alpha^{-1} = \ddagger \text{Aut}_k(U_{T_2})$ $\beta \circ \dagger \{T_2(K) \rightarrow T(K)\} \circ \alpha^{-1} = \ddagger \{T_2(K) \rightarrow T(K)\}$ PropositionLet $\square \phi, \square \lambda$. Then there exists an isom. of CFS (α, β) s.t. $(\alpha, \beta)^\dagger \phi = \ddagger \phi$,
 $(\alpha, \beta)^\dagger \lambda = \ddagger \lambda$

35. From CFS to base fields-8

Theorem 3.13, (ii)

Let $\square\mathcal{A}, \square\phi, \square\lambda, (\alpha, \beta)$

Suppose that $(\alpha, \beta)^\dagger\phi = \ddagger\phi, (\alpha, \beta)^\dagger\lambda = \ddagger\lambda$

and $(\dagger\mathcal{A}, \dagger\phi, \dagger\lambda)$ admits a CFS-admissible collection of maps

Then $(\ddagger\mathcal{A}, \ddagger\phi, \ddagger\lambda)$ admits a CFS-admissible collection of maps

Moreover, β induces a field isom. $\dagger K \xrightarrow{\sim} \ddagger K$ (cf. (i))

36. From CFS to base fields-9

Definition

$$\begin{aligned} \text{Aut}_k(U_T) &\stackrel{\text{def}}{=} \{\sigma \in \text{Aut}(U_T(K)) \mid \exists \tau \in S_5 \text{ s.t. } \tau(1) = 1, \sigma \circ \lambda = \lambda \circ \phi^{-1}(\tau)\} \\ \text{Aut}_k(U_T) &\xrightarrow{\sim} \text{Aut}(T(K) \setminus U_T(K)) \xrightarrow{\sim} S_3 \\ &\sigma \mapsto \sigma|_{T(K) \setminus U_T(K)} \end{aligned}$$

Theorem 3.13, (iii) Let $\mathcal{A}, \square\phi, \square\lambda$

Then there exists a unique element $\sigma \in \text{Aut}_k(U_T)$ s.t.

$$\sigma(\dagger 0, \dagger 1, \dagger \infty) = (\dagger 0, \dagger 1, \dagger \infty)$$

and σ determines a field isom.

$$\dagger K \xrightarrow{\sim} \dagger K$$

(cf. (ii))

37. From CFS to base fields-10

Theorem 4.8, (iii)

Let $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$\gamma: (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \xrightarrow{\sim} (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$: any isom. of PGCS

Then γ induces an isom. of CFS

$(\alpha, \beta): (T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$

$\xrightarrow{\sim} (T_2(K), T(K), T(K) \setminus U_T(K), \text{Aut}_k(U_{T_2}), \{T_2(K) \rightarrow T(K)\})$

Write

$$t_{\beta(0), \beta(1), \beta(\infty)} \in \Gamma(U_T, \mathcal{O})$$

for the regular function s.t.

- $t_{\beta(0), \beta(1), \beta(\infty)}$ induces a bijection

$$t_{\beta(0), \beta(1), \beta(\infty)}: T(K) \xrightarrow{\sim} K \sqcup \{\infty\}$$

- the zero divisor is of degree 1 and supported on $\beta(0)$
- $t_{\beta(0), \beta(1), \beta(\infty)}(\beta(1)) = 1$
- the divisor of poles is of degree 1 and supported on $\beta(\infty)$

(i.e., a functional linear transformation $x \mapsto \frac{x - \beta(0)}{x - \beta(\infty)} \frac{\beta(1) - \beta(\infty)}{\beta(1) - \beta(0)}$)

Then the bijection

$$T(K) \xrightarrow{\sim} T(K) \xrightarrow{\sim}_{t_{\beta(0), \beta(1), \beta(\infty)}} K \sqcup \{\infty\}$$

determines a field isom.

$$K \xrightarrow{\sim} K$$

that is equivariant with respect to the respective natural actions of the profinite groups $G_k^{(p)}, G_k^{(p)}$, relative to the isom. $\gamma|_G: G_k^{(p)} \xrightarrow{\sim} G_k^{(p)}$

38. From CFS to base fields-11: Summary

Theorem 3.13, (i)

If ϕ, λ : good, then $(\mathcal{A}, \phi, \lambda)$ admits CFS-adm. maps

Theorem 3.13, (ii)

Any $(\mathcal{A}, \phi, \lambda)$ admits CFS-adm. maps

Theorem 3.13, (ii)

$0, 1, \infty$ determine the field structure of K

$\exists \sigma \in \text{Aut}_k(U_T)$ s.t.

the field of $(\mathcal{A}, \dagger\phi, \dagger\lambda) \xrightarrow{\sim}_{\sigma}$ the field of $(\mathcal{A}, \dagger\phi, \dagger\lambda)$

Theorem 4.8, (iii), (Theorem 0.3)

PGCS $\mathcal{B} \rightsquigarrow K \curvearrowright G_k^{(p)}$

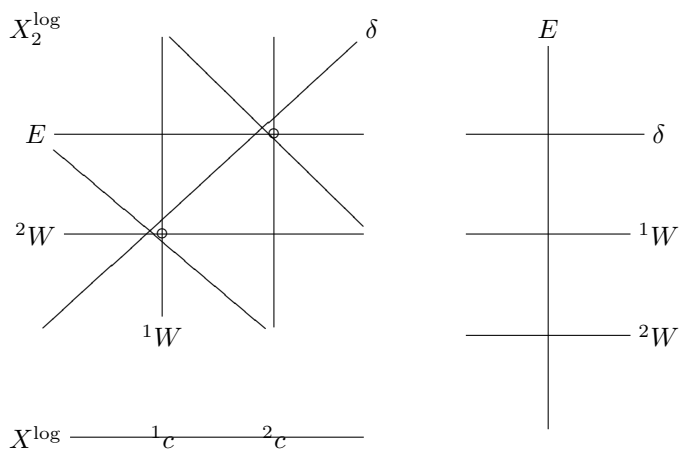
Write $k \stackrel{\text{def}}{=} K^{G_k^{(p)}}$, then PGCS $\mathcal{B} \rightsquigarrow k$

39. What are log divisors?-1

Definition

We shall refer to an irreducible divisor V of X_n contained in the complement $X_n \setminus U_{X_n}$ as a log divisor of X_n^{\log}

For example, if $r = 2$, $n = 2$

Note

- $\#LD = 7$, $\#LFS = 8$
- The inverse image of 1c is ${}^1W \cup E$
- In general, the inverse image of a log divisor of X_{n-1}^{\log} is the union of two distinct log divisors of X_n^{\log}

40. What are log divisors?-2

Note

• For a log full-point $P \in X_n^{\text{log}}$, there exist distinct log divisors ${}^1V, \dots, {}^nV$ s.t.
 $P = {}^1V \cap \dots \cap {}^nV$

• $\delta, {}^1W, {}^2W$ are isom. to X

• E is isom. to T (T^{log} is of type $(0, 3)$)

• In general, if $r > 0$, then $V^{\text{log} \leq 1}$ is isomorphic to $U_{T_m} \times_k U_{X_{n-1-m}}$ (cf. [Hgsh1], Lemma 6.1)

• (g, r, n) PGCS-collection $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$

\rightsquigarrow **LFS** (LFS $\stackrel{\text{def}}{=} \{\text{log-full subgroups}\}$) (cf. p.17)

\rightsquigarrow **LD** (LD $\stackrel{\text{def}}{=} \{\text{inertia subgroups ass'd to log divisors}\}$) (cf. [Hgsh3], Theorem A, (i))

\rightsquigarrow **GFS** (GFS $\stackrel{\text{def}}{=} \{\text{generalized fiber subgroups}\}$) (cf. p.17)

So we can consider log-full points, log divisors, and generalized fiber subgroups

41. Reconstruction of $(0, 3, 2)$ PGCS-1

Definition

We shall say that (X^{\log}, n) is tripodally very ample if one of the following conditions (i), (ii), (iii) holds:

- (i) $\#(X(k) \setminus U_X(k)) = 3$ and $(g, r, n) = (0, 3, 2)$
- (ii) $X(k) \setminus U_X(k) \neq \emptyset$, $n > 2$, and $r > 0$
- (iii) $U_X(k) \neq \emptyset$ and $n > 3$

Note

(i) ok

(ii) We consider log divisors

Since $r > 0$, $V^{\log \leq 1}$ is isomorphic to $U_{T_m} \times_k U_{X_{n-1-m}}$ (cf. [Hgsh1], Lemma 6.1)

If $n = 3$, then $\exists V$: a log divisor s.t. $V^{\log \leq 1}$ is isomorphic to U_{T_2}

\implies It seems possible?

(iii) Suppose that $r = 0$

If $n = 4$, then $\exists V$: a log divisor s.t. $V^{\log \leq 1}$ is isomorphic to $U_{T_2} \times U_X$

\implies It seems possible??

trip. very ample $\iff "T_2^{\log} \subseteq X_n^{\log}"$

42. Reconstruction of $(0, 3, 2)$ PGCS-2: More detail(ii) $c \in X(k) \setminus U_X(k)$: assumption

$$X^{\log} \longleftarrow X_2^{\log} \longleftarrow X_3^{\log} \longleftarrow X_n^{\log}$$

$$\begin{array}{ccccc} c & \cdots & U_T & \cdots & U_{T_2} \\ & \searrow & \sqcup & \searrow & \sqcup \\ & & U_X & & U_T \times_k U_X \end{array}$$

(iii) $c \in U_X(k)$: assumption

$$X_2^{\log} \longleftarrow X_3^{\log} \longleftarrow X_4^{\log} \longleftarrow X_n^{\log}$$

$$\begin{array}{ccccc} (c, c) & \cdots & U_T & \cdots & U_{T_2} \\ & \searrow & \sqcup & \searrow & \sqcup \\ & & U_X & & U_T \times_k U_X \end{array}$$

 \implies It seems possible!

43. Reconstruction of $(0, 3, 2)$ PGCS-3

(ii)

- How do we choose U_{T_2} from $\{U_{T_2}, U_T \times_k U_X\}$?

Since $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$ is indecomposable, and $\pi_1(U_T \times_k \bar{k})^{(p)} \times \pi_1(U_X \times_k \bar{k})^{(p)}$ is decomposable (cf. [Hgsh1], §6; [Ind], Theorem 3.5)

So we choose U_{T_2}

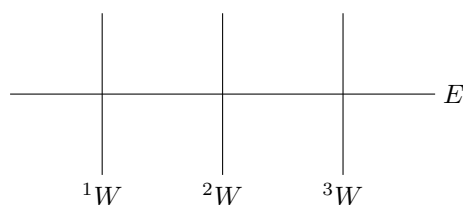
- How do we choose U_T from $\{U_T, U_X\}$? Since $\pi_1(U_T \times_k \bar{k})^{(p)}$ is isom. to $\pi_1(U_X \times_k \bar{k})^{(p)} \iff (g, r) = (0, 3), (1, 1)$

(Note that U_X is hyperbolic curve of type (g, r))

If the case $(g, r) = (0, 3)$, there's nothing to do.

So we consider $(g, r) = (1, 1)$ and $(g, r, n) = (1, 1, 2)$

Since $(g, r, n) = (1, 1, 2)$, then $\#LD = 4$



${}^1W^{\log \leq 1}$ is isom. to U_X

$E^{\log \leq 1}$ is isom. to U_T

So we choose E and U_T

44. Reconstruction of $(0, 3, 2)$ PGCS-4: Summary

START: trip. very ample (g, r, n) PGCS-collection $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$

\rightsquigarrow LD, GFS (cf. p.40)

Thus, \rightsquigarrow trip. very ample (g, r, m) PGCS-collection $(\pi_1(U_{X_m})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_m})$

\rightsquigarrow $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

(cf. [Hgsh2], Remark 6.3; [NodNon], Theorem A)

45. Recent research-1

trip. very ample $\iff T_2^{\log} \subseteq X_n^{\log}$

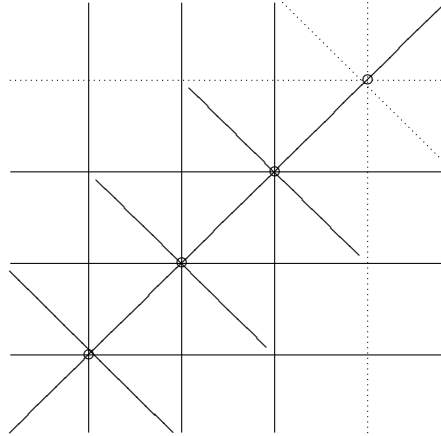
In recent research, the reconstruction can be applied in situation “ $T_2^{\log} \supseteq X_n^{\log}$ ”,
e.g.,

$(0, r, 2)$ PGCS-collection $(\pi_1(U_{X_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_2})$ ($r \geq 3$)

$\rightsquigarrow (0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

Since the difference between T_2^{\log} and X_2^{\log} is a few log divisors (cf. [Hgsh3], §2)

46. Recent research-2



the number of log divisors of $(g, r, n) = (0, 3, 2)$ is 10 (all lines)

the number of log divisors of $(0, 4, 2)$ is 13 (all lines and all dotted lines)

The difference is only three dotted lines, so it seems possible

$(0, 4, 2)$ PGCS-collection $(\pi_1(U_{X_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_2})$

$\rightsquigarrow (0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

47. Semi-absolute bi pro- p GC-1

Theorem 0.1

$\square k$: a generalized sub- p -adic field

$\square \mathcal{B} \stackrel{\text{def}}{=} (\square \pi_1(U_{X_n})^{(p)}, \square G_k^{(p)}, \square \mathcal{D}_{X_n})$: a trip. very ample $(\square g, \square r, \square n)$ PGCS-collection

Then the natural morphism

$$\text{Isom}({}^\dagger U_{X_n}, {}^\ddagger U_{X_n}) \rightarrow \text{Isom}({}^\dagger \mathcal{B}, {}^\ddagger \mathcal{B})/\text{Inn}$$

is bijective

proof. First, we consider the following Claim

Claim

PGCS-collection $\mathcal{B} \rightsquigarrow (g, r, n)$

(pf.) There exist two reconstruction.

- [HMM], Theorem A, (i),
- [Hgsh3], Theorem A, (ii), (if $r > 0$) (using LFS)

by Claim, write $(g, r, n) \stackrel{\text{def}}{=} (\square g, \square r, \square n)$

48. Semi-absolute bi pro- p GC-2: proof of Claim

By [Hgsh3]

- $\mathcal{B} \rightsquigarrow \text{LFS} \rightsquigarrow \text{LD}$

- a log-full subgroup is isom. to $\mathbb{Z}_p^{\oplus n}$ (cf. p.17)

$\rightsquigarrow n$

- $\#\text{LFS} = \prod_{i=0}^{n-1} (r + 2i)$

For example, if $r = 2$, $n = 2$, then $\#\text{LFS} = 2 \times 4 = 8$ (cf. p.39)

$\rightsquigarrow r$

- $\text{LD} \rightsquigarrow \{\pi_1(U_{T_m} \times_k \bar{k})^{(p)}, \pi_1(U_{X_m} \times_k \bar{k})^{(p)}\}_{m=1}^{n-1}$

$\rightsquigarrow \{\pi_1(U_T \times_k \bar{k})^{(p)}, \pi_1(U_X \times_k \bar{k})^{(p)}\}$

Note that $\pi_1(U_{X_m} \times_k \bar{k})^{(p)}$ is elastic $\iff m = 1$

- Write $N(g, r)$ for the number of generators of $\pi_1(U_X \times_k \bar{k})^{(p)}$

Note that $N(g, r) = 2g + r - 1 \geq N(0, 3) = 2$

i.e., $\max(N(g, r), N(0, 3)) = 2g + r - 1$

$$\rightsquigarrow g \stackrel{\text{def}}{=} \frac{\max(N(g, r), N(0, 3)) - r + 1}{2}$$

This completes the proof of Claim

(g, r, n) PGCS-collection $\mathcal{B} \rightsquigarrow (g, r, n)$ if $r > 0$

49. Semi-absolute bi pro- p GC-3

Since $\square G_k^p$ is center-free (cf. [Hgsh2], Lemma 5.3; [LocAn], Lemma 15.8)

$$\text{Isom}({}^\dagger U_{X_n}, {}^\ddagger U_{X_n}) \rightarrow \text{Isom}({}^\dagger \mathcal{B}, {}^\ddagger \mathcal{B})/\text{Inn}$$

is injective

Using

trip. very ample (g, r, n) PGCS-collection $\mathcal{B} \rightsquigarrow (0, 3, 2)$ PGCS-collection $\rightsquigarrow k$
and pro- p version of [Topics], Theorem 4.12 (relative bi)

$$\text{Isom}({}^\dagger U_{X_n}, {}^\ddagger U_{X_n}) \rightarrow \text{Isom}({}^\dagger \mathcal{B}, {}^\ddagger \mathcal{B})/\text{Inn}$$

is bijective

50. Reconstruct function fields-1

START: $(0, 3, 2)$ PGCS-collection $\mathcal{B} = (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$ and $\dagger\pi_1(U_{T_2})$

By Mochizuki's mono profinite GC

$\dagger\pi_1(U_{T_2}) \rightsquigarrow \dagger\pi_1(U_T), \dagger G_k, \text{Funct}(Z), \dagger\bar{k}, \dagger\mathcal{B} = (\dagger\pi_1(U_{T_2})^{(p)}, \dagger G_k^{(p)}, \dagger\mathcal{D}_{T_2})$

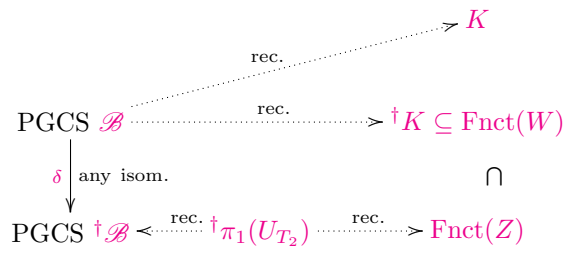
where Z denotes the profinite étale covering of U_T s.t. $\pi_1(U_T) = \text{Gal}(Z/U_T)$

Let $\delta: \mathcal{B} \xrightarrow{\sim} \dagger\mathcal{B}$ isom. of PGCS and $E \in \text{GFS}$ (cf. p.17)

$$\begin{array}{ccccc}
 \pi_1(U_{T_2})^{(p)} & \longrightarrow & \pi_1(U_T)^{(p)} & \longrightarrow & G_k^{(p)} \\
 \delta \downarrow \simeq & & \uparrow & & \uparrow \\
 \dagger\pi_1(U_{T_2})^{(p)} & \longleftarrow & \dagger\pi_1(U_{T_2}) & \longrightarrow & \dagger\pi_1(U_T) & \longrightarrow & \dagger G_k \\
 & & \text{Funct}(W) \stackrel{\text{def}}{=} \text{Funct}(Z)^{\text{Ker}(\dagger\pi_1(U_T) \rightarrow \pi_1(U_T)^{(p)})} & & & & \\
 & & \dagger K \stackrel{\text{def}}{=} \dagger\bar{k}^{\text{Ker}(\dagger G_k \rightarrow G_k^{(p)})} & & & &
 \end{array}$$

where W denotes the profinite étale covering of U_T s.t. $\pi_1(U_T)^{(p)} = \text{Gal}(W/U_T)$

51. Reconstruct function fields-2



Next, we consider K vs. $\dagger K \subseteq \text{Funct}(W)$

52. Reconstruct function fields-3

Let $T \in \text{Funct}(W)$, then T induces a map

$$T: \mathcal{D}_T \rightarrow {}^\dagger K \sqcup \{\infty\}$$

There exists a unique element $T \in \text{Funct}(U_X) \stackrel{\text{def}}{=} \text{Funct}(W)^{\pi_1(U_X)^{(p)}}$ s.t.

- the zero divisor of T is of degree 1 and supported on 0
- $T(1) = {}^\dagger 1$
- the divisor of poles of T is of degree 1 and supported on ∞

then T induces a field isom.

$$K \xrightarrow{\sim} {}^\dagger K$$

(Theorem 0.4)

53. Reconstruct function fields-4

START: $(0, 3, 2)$ PGCS-collection $\mathcal{B} = (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$ and $\dagger\pi_1(U_{T_2})$

Let $\delta: \mathcal{B} \xrightarrow{\sim} \dagger\mathcal{B}$: isom. of PGCS

$E \in \text{GFS}$

Then $\rightsquigarrow K \xrightarrow{\sim} \dagger K \subseteq \text{Funct}(W)$ (Theorem 0.4)

Note that $\#\text{GFS} = \#\{\text{generalized fiber subgroups}\} = 5$

$$E_{\cap} \stackrel{\text{def}}{=} \bigcap_{E \in \text{GFS}} E, \quad \pi_1(U_{T_2})_{2 \rightarrow 1}^{(p)} \stackrel{\text{def}}{=} \pi_1(U_{T_2})^{(p)} / E_{\cap}$$

$$\text{Funct}(W) \hookrightarrow \text{RatMaps}(\mathcal{D}_T, K): T \mapsto T(-)$$

$$\text{Funct}(W) \hookrightarrow \text{RatMaps}(\mathcal{D}_{T_2}, K)$$

$$T \mapsto T(-) \circ p_{2/1}$$

The field structure of K induces a natural ring structure on $\text{RatMaps}(\mathcal{D}_{T_2}, K)$

54. Reconstruct function fields-5

We write R for the subring of $\text{RatMaps}(\mathcal{D}_{T_2}, K)$ generated by the subrings

$$\text{Funct}(W) \hookrightarrow \text{RatMaps}(\mathcal{D}_{T_2}, K)$$

for $E \in \text{GFS}$

Then R is an integral domain on which the subgroup $E_\cap \subseteq \pi_1(U_{T_2})^{(p)}$ acts trivially

The quotient field $\text{Funct}(Y) \stackrel{\text{def}}{=} \text{Frac}(R)$ which is equipped with an action by $\pi_1(U_{T_2})_{2 \rightarrow 1}^{(p)}$

where $Y \rightarrow U_{T_2}$ for the profinite étale covering corresponding to $\pi_1(U_{T_2})_{2 \rightarrow 1}^{(p)}$ (Theorem 0.2)

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